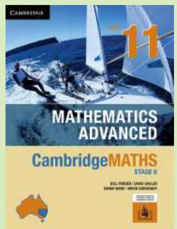
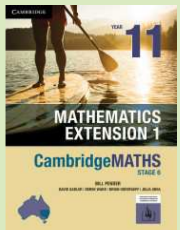
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

With references to





Ⓐ **Trigonometry**

Name: ....

Initial version by H. Lam, February 2015. Last updated March 8, 2021
Various corrections by students and members of the Department of Mathematics at North Sydney Boys High School and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under (cc) CC BY 2.0.

Parts of this document are also sourced from:

- Exercises on page 6: Jones and Couchman (1981, Ex 8.1)
- Exercises on pages 18, 11 and 15: Grove (2010, Ex 6.7)

Symbols used:

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

(2) Mathematics (2 Unit) legacy course content/textbook

(A) Mathematics Advanced content/textbook

(X1) Mathematics Extension 1 content/textbook

Extension work.

Memorisation required.

Enrichment & problem solving.

Understanding (as opposed to blatant memorisation) is required.

Warning. Beware!

Available on NESA Reference Sheet

Syllabus outcomes addressed

MA11-3 uses the concepts and techniques of trigonometry in the solution of equations and problems involving geometric shapes

MA11-4 uses the concepts and techniques of periodic functions in the solutions of trigonometric equations or proof of trigonometric identities

Syllabus subtopics

MA-T1 Trigonometry and Measure of Angles

MA-T2 Trigonometric Functions and Identities

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from (2) *Cambridge Year 11 2 Unit* or (X1) *Cambridge Year 11 3 Unit* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book, unless it is a worded problem!

Contents

Section 1

Trigonometric Ratios

Learning Goal(s)

Knowledge
Three trigonometric ratios in right angled triangles

Skills
Solve simple problems involving right angled triangles

Understanding
How sine, cosine and tangent are interrelated

By the end of this section am I able to:

4.1 Use the sine, cosine and tangent ratios to solve problems involving right-angled triangles where angles are measured in degrees, or degrees and minutes

4.2 Define the reciprocal trigonometric functions

4.12 Use $\tan x = \dfrac{\sin x}{\cos x}$ provided that $\cos x \neq 0$

4.13 Evaluate trigonometric expressions using angles of any magnitude and complementary angle results

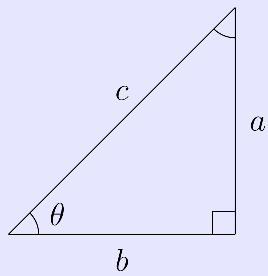
Definition 1

In relation to the angle marked θ,

$\sin\theta = \_\_\_\_$ $\qquad \cos(90^\circ - \theta) = \_\_\_\_$

$\cos\theta = \_\_\_\_$ $\qquad \sin(90^\circ - \theta) = \_\_\_\_$

$\tan\theta = \_\_\_\_ = \_\_\_\_$



1.1 Relationship for sine/cosine

Important note

"Co" in the **co**sine indicates the ....complement........ of sine.

Example 1

Find the value of x:

(a) $\cos x^\circ = \sin 70^\circ$. (b) $\sin x^\circ = \cos 55^\circ$.

Example 2

Find the value of x:

(a) $\cos(x-20)^\circ = \sin 40^\circ$.

(b) $\sin 2x^\circ = \cos 10^\circ$.

Example 3

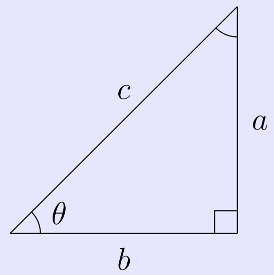
Fully simplify $\dfrac{3\sin 75^\circ}{\cos 15^\circ}$ without using a calculator.

1.2 The reciprocal ratios

Definition 2

In relation to the angle marked θ,

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$



Important note

"Co" in the **co**secant/cotangent indicates the ......complement...... of secant/tangent.

Example 4

From the triangle above, express in terms of a, b and c:

(a) $\operatorname{cosec}\theta =$ ——

(b) $\operatorname{cosec}(90^\circ-\theta) =$ ——

(c) $\cot\theta =$ ——

(d) $\sec(90^\circ-\theta) =$ ——

(e) $\sec\theta =$ ——

(f) $\cot(90^\circ-\theta) =$ ——

Exercises

Source: Jones and Couchman (1981, Ex 8.1)

1. Find the value of x if

(a) $\operatorname{cosec} x^\circ = \sec 60^\circ$

(b) $\cot x = \tan 40^\circ$

(c) $\sec x = \operatorname{cosec} 35^\circ$

(d) $\tan x = \cot 65^\circ$

2. Without using a calculator, evaluate:

(a) $\dfrac{\sin 70^\circ}{\cos 20^\circ}$.

(b) $\dfrac{\operatorname{cosec} 40^\circ}{\sec 50^\circ}$.

(c) $\dfrac{\tan 80^\circ}{\cot 10^\circ}$.

3. Find the value of x if

(a) $\tan 20^\circ = \cot(x+30)^\circ$

(b) $\sin x^\circ = \cos(x + 50^\circ)$

Answers

1. (a) 30 (b) 50 (c) 55 (d) 25 **2.** (a) 1 (b) 1 (c) 1 **3.** (a) 40 (b) 20

Further exercises

(A) **Ex 5A**

- Q1, 4, 5, 8, 9, 12, 13
- Q15-18

(X1) **Ex 6A**

- Q1-5 last row
- Q11-13 last column

1.3 Problem solving with right angled trigonometry

Important note

⚠ Draw picture!

Example 5

A plane flying at 400 km/h flies from A to B in a direction S30°E for 15 minutes, then turns sharply to fly due east for 30 minutes to C.

(a) Find how far south and east of A the point B is.

(b) Find the true bearing of C from A, to the nearest degree.

Answer: (a) $50\sqrt{3}$ km south, 50 km east (b) 109°T

Example 6

A walker walks on a flat plane directly towards a distant high rocky outcrop R. At point A the angle of elevation of the outcrop is 24°, and a kilometre closer at B the angle of elevation is 32°.

(a) Find the horizontal distance from B to the outcrop, to the nearest metre.

(b) Find the height of the outcrop above the plane, to the nearest metre.

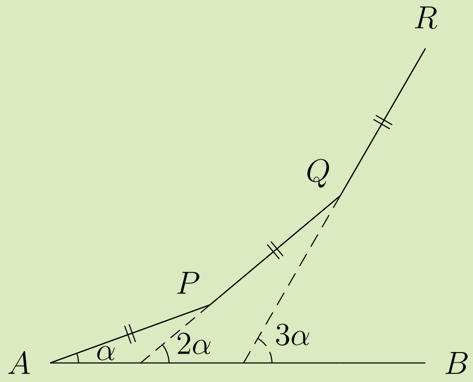
Answer: (a) 2.478 (b) 1.549

Example 7

[Ex 4B Q9] (Pender, Sadler, Shea, & Ward, 1999)

⚠ AP, PQ and QR are three equal intervals inclined at angles α, 2α and 3α respectively to interval AB. Show that

$$\tan\angle BAR = \frac{\sin\alpha + \sin 2\alpha + \sin 3\alpha}{\cos\alpha + \cos 2\alpha + \cos 3\alpha}$$



Further exercises

Ⓐ Ex 5A

- Q7, 11-13 last 2 columns

Ⓐ Ex 5B

- Q1-17

(X1) Ex 6A

- Q18-21

(X1) Ex 6B

- Q1-20

Learning Goal(s)

Knowledge
Exact ratios for particular special angles

Skills
Perform calculations based on exact ratios and values

Understanding
Unit circle definitions of trigonometric ratios

By the end of this section am I able to:

4.3 Understand the unit circle definition of $\sin\theta$, $\cos\theta$ and $\tan\theta$ and periodicity using degrees.

4.4 Evaluate trigonometric expressions using angles of any magnitude

4.5 Given one trigonometric function, find another

1.4 Exact values and angles of any magnitude

This exact values table is second only to the multiplication tables!

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

Example 8

Find $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ if $\alpha =$

1. 30° **3.** 45° **5.** 150° **7.** 240° **9.** 330°

2. 60° **4.** 90° **6.** 225° **8.** 315° **10.** 300°

Example 9

Evaluate, without using a calculator:

(a) $\tan 30^\circ \sin 60^\circ$

(b) $\tan^2 60^\circ - \sin^2 60^\circ$

Example 10

Find the exact value, giving answers in simplest surd form with a rational denominator.

(a) $\operatorname{cosec} 30^\circ + \cot 45^\circ$

(b) $\dfrac{\sin 45^\circ}{\sec^2 60^\circ}$

Exercises

1. Find all quadrants where

(a) $\cos\theta > 0$
(b) $\tan\theta > 0$
(c) $\sin\theta > 0$
(d) $\tan\theta < 0$
(e) $\sin\theta < 0$
(f) $\cos\theta < 0$
(g) $\sin\theta < 0$ & $\cos\theta < 0$
(h) $\sin\theta < 0$ & $\tan\theta > 0$
(i) $\cos\theta < 0$ & $\tan\theta > 0$
(j) $\sin\theta > 0$ & $\tan\theta > 0$

2. (a) Which quadrant is the angle $240°$ in? (b) Find the exact value of $\cos 240°$.

3. (a) Which quadrant is the angle $315°$ in? (b) Find the exact value of $\sin 315°$.

4. (a) Which quadrant is the angle $120°$ in? (b) Find the exact value of $\tan 120°$.

5. (a) Which quadrant is the angle $-225°$ in?
(b) Find the exact value of $\sin -225°$.

6. (a) Which quadrant is the angle $-330°$ in?
(b) Find the exact value of $\cos -330°$.

7. Find the exact value of each ratio:

(a) $\tan 225°$
(b) $\cos 315°$
(c) $\tan 300°$
(d) $\sin 150°$
(e) $\cos 120°$
(f) $\sin 210°$
(g) $\cos 330°$
(h) $\tan 150°$
(i) $\sin 300°$
(j) $\cos 135°$

8. Find the exact value of each ratio:

(a) $\cos(-225°)$
(b) $\cos(-210°)$
(c) $\tan(-300°)$
(d) $\cos(-150°)$
(e) $\sin(-60°)$
(f) $\tan(-240°)$
(g) $\cos(-300°)$
(h) $\tan(-30°)$
(i) $\cos(-45°)$
(j) $\sin(-135°)$

9. Find the exact value of each ratio:

(a) $\cos 570°$
(b) $\tan 420°$
(c) $\sin 480°$
(d) $\cos 660°$
(e) $\sin 690°$
(f) $\tan 600°$
(g) $\sin 495°$
(h) $\cos 405°$
(i) $\tan 675°$
(j) $\sin 390°$

Source: Grove (2010, Ex 6.7)

Answers

1. (a) $1, 4$ (b) $1, 3$ (c) $1, 2$ (d) $2, 4$ (e) $3, 4$ (f) $2, 3$ (g) 3 (h) 3 (i) 3 (j) 1 **2.** (a) 3 (b) $-\frac{1}{2}$ **3.** (a) 4 (b) $-\frac{1}{\sqrt{2}}$ **4.** (a) 2 (b) $-\sqrt{3}$ **5.** (a) 2 (b) $\frac{1}{\sqrt{2}}$ **6.** (a) 1 (b) $\frac{\sqrt{3}}{2}$ **7.** (a) 1 (b) $\frac{1}{\sqrt{2}}$ (c) $-\sqrt{3}$ (d) $\frac{1}{2}$ (e) $-\frac{1}{2}$ (f) $-\frac{1}{2}$ (g) $\frac{\sqrt{3}}{2}$ (h) $-\frac{1}{\sqrt{3}}$ (i) $-\frac{\sqrt{3}}{2}$ (j) $-\frac{1}{\sqrt{2}}$ **8.** (a) $-\frac{1}{\sqrt{2}}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) $-\frac{\sqrt{3}}{2}$ (e) $-\frac{\sqrt{3}}{2}$ (f) $-\sqrt{3}$ (g) $\frac{1}{2}$ (h) $-\frac{1}{\sqrt{3}}$ (i) $\frac{1}{\sqrt{2}}$ (j) $-\frac{1}{\sqrt{2}}$ **9.** (a) $-\frac{\sqrt{3}}{2}$ (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$ (e) $-\frac{1}{2}$ (f) $\sqrt{3}$ (g) $\frac{1}{\sqrt{2}}$ (h) $\frac{1}{\sqrt{2}}$ (i) -1 (j) $\frac{1}{2}$

Further exercises

(A) Ex 5D
- Q1-2, 6

(A) Ex 5E
- All questions

(X1) Ex 6D
- Q1-2, 6

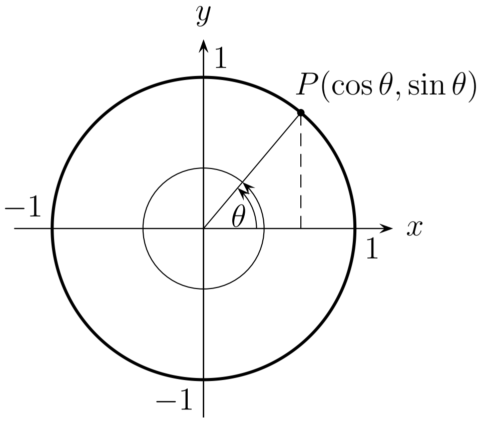
(X1) Ex 6E
- All questions

1.5 Other ratios

Laws/Results

$$\sin(360^\circ + \theta) = \underline{\sin\theta} \qquad \cos(360^\circ + \theta) = \underline{\cos\theta}$$

$$\tan(360^\circ + \theta) = \underline{\tan\theta}$$

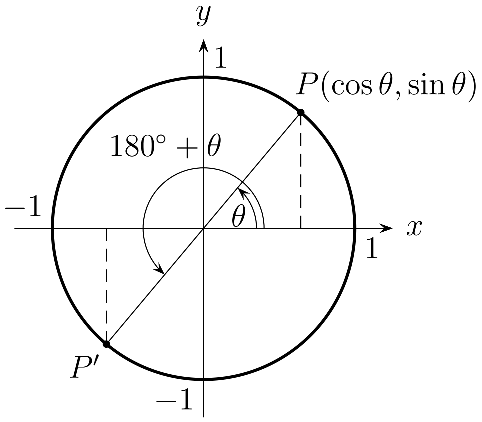


- Adding one ......revolution...... will not alter the ratios.

Laws/Results

$$\sin(180^\circ + \theta) = \underline{-\sin\theta} \qquad \cos(180^\circ + \theta) = \underline{-\cos\theta}$$

$$\tan(180^\circ + \theta) = \underline{\tan\theta}$$

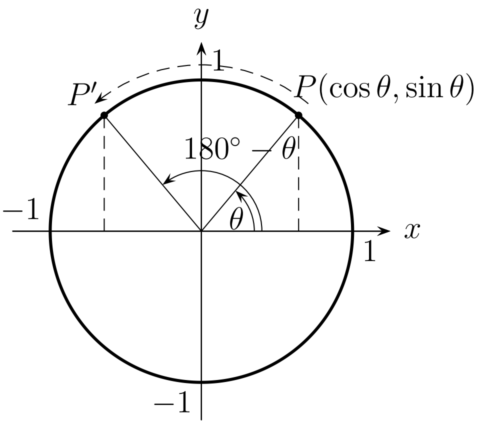


- Adding 180° places P into the ......opposite...... quadrant.
- All ratios except ..tan.. changes signs.

Laws/Results

$$\sin(180^\circ - \theta) = \underline{\sin\theta} \qquad \cos(180^\circ - \theta) = \underline{-\cos\theta}$$

$$\tan(180^\circ - \theta) = \underline{-\tan\theta}$$

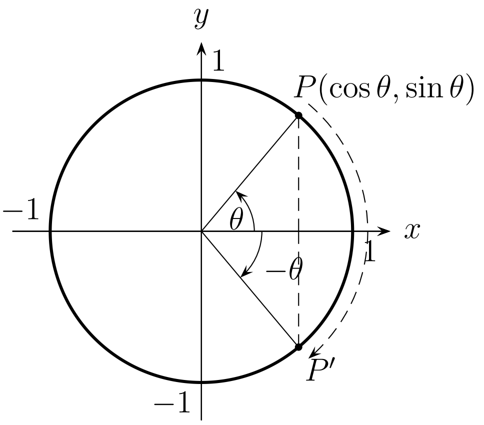


- Subtracting from 180° places P into the horizontally adjacent quadrant.
- Only sin does not changes signs.

Laws/Results

$$\sin(-\theta) = \underline{-\sin\theta} \qquad \cos(-\theta) = \underline{\cos\theta}$$

$$\tan(-\theta) = \underline{-\tan\theta}$$



- Negating θ places P into the vertically adjacent quadrant.
- Only cosine does not change signs.

Exercises

Simplify fully:

1. $\sin(180^\circ - \theta)$

2. $\cos(360^\circ - x)$

3. $\tan(180^\circ + \alpha)$

4. $\sin(180^\circ + \alpha)$

5. $\tan(360^\circ - \theta)$

6. $\sin(-\theta)$

7. $\cos(-\alpha)$

8. $\tan(-x)$

Source: Grove (2010, Ex 6.7) .

Answers

1. $\sin\theta$ **2.** $\cos x$ **3.** $\tan\alpha$ **4.** $-\sin\alpha$ **5.** $-\tan\theta$ **6.** $-\sin\theta$ **7.** $\cos\alpha$ **8.** $-\tan x$

1.6 Given one ratio, find another

Important note

- Draw correct diagram depicting angle in the appropriate quadrant.
- Do NOT use the calculator to evaluate the pronumeral!

Example 11

Given $\sec\theta = 3$ and θ is acute, find the value of $\cos\theta$, $\tan\theta$ and $\sin\theta$.

Steps

1. Draw correct diagram depicting $\sec\theta = 3$ and acute θ in the correct quadrant:

2. Find missing side length (label on diagram).

3. Evaluate the other ratios:

Example 12

Find $\sin x$ and $\tan x$ in exact surd form when $\cos x = -\frac{2}{3}$ and $90° < x < 180°$.

Example 13

Given $\sin\theta = \frac{3}{7}$ and $\cos\theta < 0$, evaluate $\cos\theta$ and $\tan\theta$.

Example 14

Given $\sin\theta = -\frac{3}{8}$ and $\tan\theta > 0$, find the value of $\cos\theta$ and $\cot\theta$.

Example 15

If $\tan x = \frac{3}{2}$ and $180^\circ < x < 270^\circ$, find the value of $\operatorname{cosec} x$ and $\cos x$.

Exercises

1. If $\sin\theta = \frac{4}{7}$ and $\tan\theta < 0$, find the exact value of $\cos\theta$ and $\tan\theta$.
2. If $\sin x < 0$ and $\tan x = -\frac{5}{8}$, find the exact value of $\cos x$ and $\operatorname{cosec} x$.
3. Given $\cos x = \frac{2}{5}$ and $\tan x < 0$, find the exact value of $\operatorname{cosec} x$, $\cot x$ and $\tan x$.
4. If $\cos x < 0$ and $\sin x < 0$, find $\cos x$ and $\sin x$ in surd form with a rational denominator if $\tan x = \frac{5}{7}$.
5. If $\sin\theta = -\frac{4}{9}$ and $270° < \theta < 360°$, find the exact value of $\tan\theta$ and $\sec\theta$.
6. If $\cos\theta = -\frac{3}{8}$ and $180° < \theta < 270°$, find exact values of $\tan x$, $\sec x$ and $\operatorname{cosec} x$.
7. Given $\sin x = 0.3$ and $\tan x < 0$,
   (a) Express $\sin x$ as a fraction.
   (b) Find the exact value of $\cos x$ and $\tan x$.
8. If $\tan\alpha = -1.2$ and $270° < \theta < 360°$, find the exact values of $\cot\alpha$, $\sec\alpha$ and $\operatorname{cosec}\alpha$.
9. Given that $\cos\theta = -0.7$, and $90° < \theta < 180°$, find the exact value of $\sin\theta$ and $\cot\theta$.

Source: Grove (2010, Ex 6.7)

Answers

1. $\cos\theta = -\frac{\sqrt{33}}{7}$, $\tan\theta = -\frac{4}{\sqrt{33}}$ **2.** $\cos x = \frac{8}{\sqrt{89}}$, $\operatorname{cosec} x = -\frac{\sqrt{89}}{5}$ **3.** $\operatorname{cosec} x = -\frac{5}{\sqrt{21}}$, $\cot x = -\frac{2}{\sqrt{21}}$, $\tan x = -\frac{\sqrt{21}}{2}$ **4.** $\cos x = -\frac{7\sqrt{74}}{74}$, $\sin x = -\frac{5\sqrt{74}}{74}$ **5.** $\tan\theta = -\frac{4}{\sqrt{65}}$, $\sec\theta = \frac{9}{\sqrt{65}}$ **6.** $\tan x = \frac{\sqrt{55}}{3}$, $\sec x = -\frac{8}{3}$, $\operatorname{cosec} x = -\frac{8}{\sqrt{55}}$ **7.** (a) $\sin x = \frac{3}{10}$ (b) $\cos x = -\frac{\sqrt{91}}{10}$, $\tan x = -\frac{3}{\sqrt{91}}$ **8.** $\cos\alpha = -\frac{5}{6}$, $\sec\alpha = \frac{\sqrt{61}}{5}$, $\operatorname{cosec}\alpha = -\frac{\sqrt{61}}{6}$ **9.** $\sin\theta = \frac{\sqrt{51}}{10}$, $\cot\theta = -\frac{7}{\sqrt{51}}$.

Further exercises

(A) **Ex 5F**
- All questions

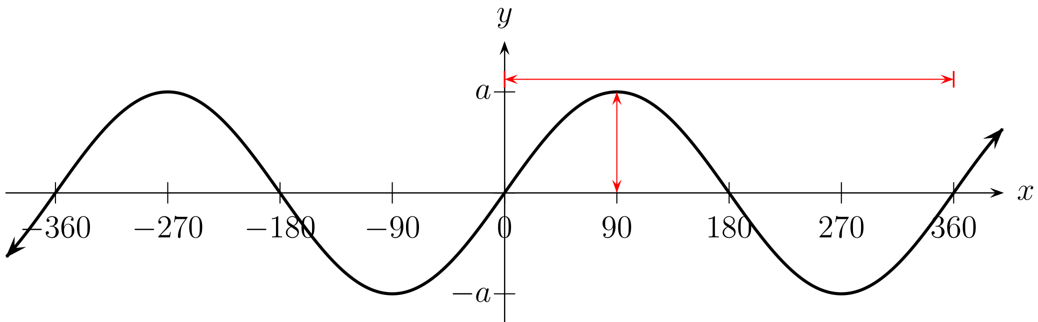
(X1) **Ex 6F**
- All questions

1.7 Trigonometric Graphs

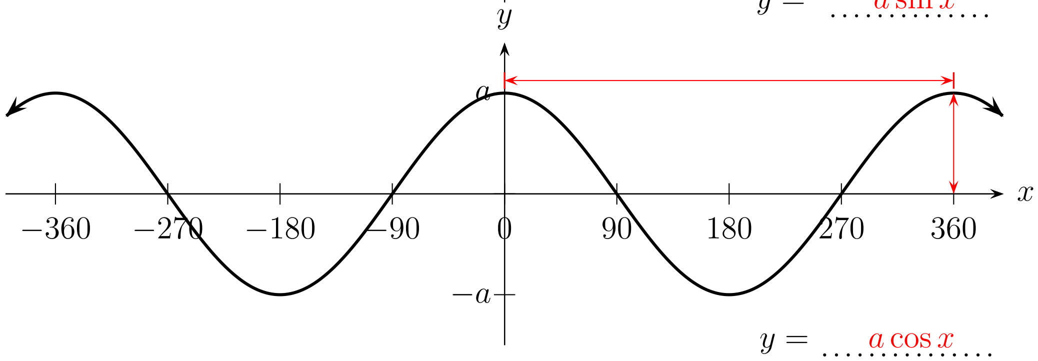
Definition 3

For $y = a \sin nx$ and $y = a \cos nx$:

- ......Amplitude...... : distance between mean (equilibrium) position & peak/trough.
  Symbol: ..a.. .
- ......Frequency...... : number of complete appearances between $0°$ and $360°$.
  Symbol: ..n.. .
- ....Period.... : the number of degrees before the graph repeats itself.
  Relationship: .... $T = \dfrac{360°}{n}$ .... .



$y =$ .... $a \sin x$ ....



$y =$ .... $a \cos x$ ....

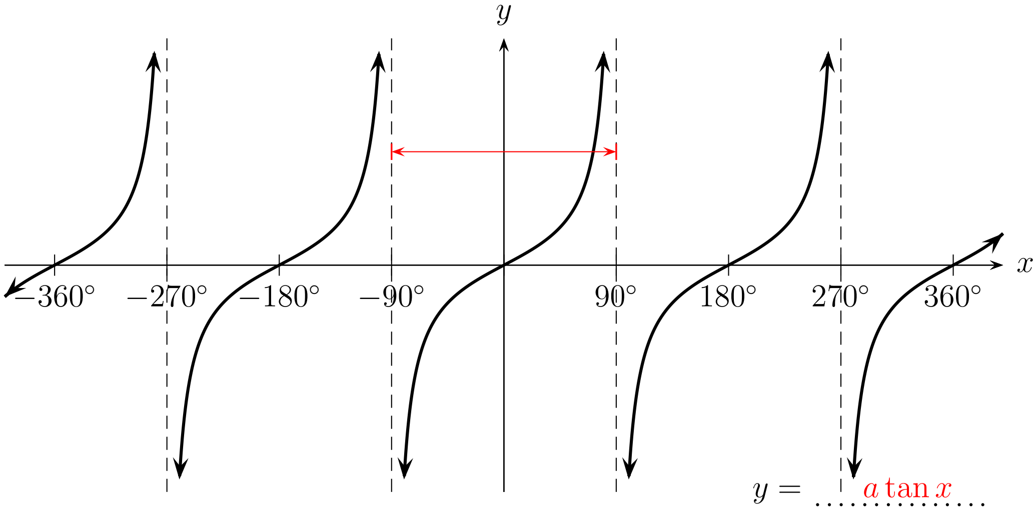
GeoGebra

frequency - amplitude.ggb

Definition 4

For $y = a\tan nx$:

- Vertical stretch factor.
  Symbol: a
- Frequency : number of complete appearances between $-90°$ and $90°$.
  Symbol: n .
- Period : the number of degrees before the graph repeats itself.
  Relationship: $T = \dfrac{180°}{n}$.



Example 16

Sketch $y = 5\cos 4x$, $-180^\circ \leq x \leq 180^\circ$.

Example 17

Sketch $y = 3\tan 2x$, $-180^\circ \leq x \leq 180^\circ$.

Example 18

Sketch $y = \frac{1}{4}\sin \frac{1}{3}x$, $-540^\circ \leq x \leq 540^\circ$.

Exercises

1. Draw the graph of $y = 2\sin x$, $-360^\circ \leq x \leq 360^\circ$. State the amplitude and period.

2. Draw the graph of $y = 4\cos x$, $-180^\circ \leq x \leq 180^\circ$. State the amplitude and period.

3. Draw the graph of $y = \tan x$, $0^\circ \leq x \leq 360^\circ$.

4. Find the periods and amplitude (where necessary) of the following:

(a) $y = 3\sin 4x$ (b) $y = 5\cos 3x$ (c) $y = \tan 4x$ (d) $y = \tan 2x$

5. Sketch the following graphs:

(a) $y = \sin 2x$, $0^\circ \leq x \leq 180^\circ$

(b) $y = \cos 3x$, $0^\circ \leq x \leq 120^\circ$

(c) $y = \sin 3x$, $0^\circ \leq x \leq 360^\circ$

(d) $y = \cos 4x$, $0^\circ \leq x \leq 180^\circ$

(e) $y = \sin \dfrac{1}{2}x$, $0^\circ \leq x \leq 360^\circ$

(f) $y = \tan \dfrac{1}{2}x$, $0^\circ \leq x \leq 180^\circ$

(g) $y = \sin \dfrac{1}{3}x$, $0^\circ \leq x \leq 270^\circ$

(h) $y = \cos \dfrac{1}{3}x$, $0^\circ \leq x \leq 540^\circ$

(i) $y = \tan \dfrac{2}{3}x$, $-135^\circ \leq x \leq 135^\circ$

(j) $y = 3\cos x$, $0^\circ \leq x \leq 180^\circ$

(k) $y = 5\cos x$, $0^\circ \leq x \leq 360^\circ$

(l) $y = -\cos x$, $0^\circ \leq x \leq 360^\circ$

(m) $y = -3\cos x$, $0^\circ \leq x \leq 360^\circ$

(n) $y = -2\sin x$, $0^\circ \leq x \leq 360^\circ$

Section 2

Trigonometric identities and equations

Learning Goal(s)

Knowledge
Transformations between $\sin^2\theta$ to $\cos^2\theta$ and $\sec^2\theta$ to $\tan^2\theta$

Skills
Manipulate expressions/solve equations involving these Pythagorean identities

Understanding
Difference between simplifying a trigonometric expression versus solving a trigonometric equation

By the end of this section am I able to:

4.10 Know the difference between an equation and an identity

4.11 Prove and apply the Pythagorean identities

4.14 Prove trigonometric identities

4.15 Simplify trigonometric expressions and solve trigonometric equations, including those that reduce to quadratic equations

2.1 Pythagorean identity

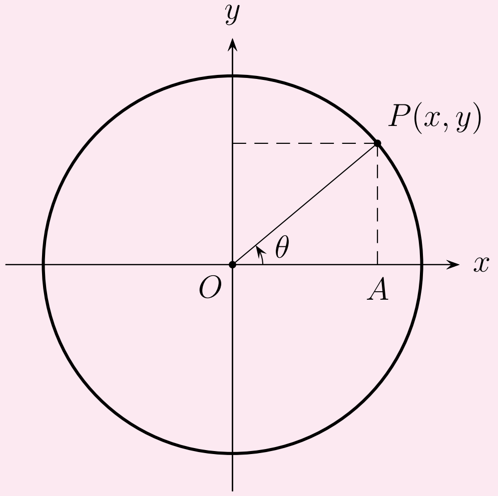
Definition 5

The Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Steps

Derivation: draw circle of radius r, centred at origin.



1. Find the lengths of OA, OP and AP:

- $OA = x$
- $OP = r$
- $AP = y$

2. Relate OA, AP and OP: $x^2 + y^2 = r^2$

3. Relate OP, AP and θ: $\dfrac{AP}{OP} = \sin\theta$

4. Relate OP, OA and θ: $\dfrac{OA}{OP} = \cos\theta$

2. Write in terms of θ only:

$$r^2 \sin^2\theta + r^2 \cos^2\theta = r^2$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad (\ddagger)$$

3. Divide $(\ddagger)$ by $\cos^2\theta$:

$$1 + \tan^2\theta = \sec^2\theta$$

4. Divide $(\ddagger)$ by $\sin^2\theta$:

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Example 19

Fully simplify: $1 - \cot^2\theta + \operatorname{cosec}^2\theta$.

Example 20

Fully simplify: $\cot^2\theta - \dfrac{1}{\sin^2\theta}$.

Example 21

Prove that $\sin^2\theta + \tan^2\theta = \sec^2\theta - \cos^2\theta$.

Example 22

Prove that $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos^2\theta - \sin^2\theta$.

Further exercises

(A) Ex 5G

- Q3-8 last 2 columns
- Q11-13

(X1) Ex 6G

- Q4-13
- Q15-16

2.1.1 Elimination of θ

Steps

1. Change subject to $\sin\theta$ or $\cos\theta$, whichever is appropriate.
2. Use Pythagorean Identity to remove θ.

Example 23

Eliminate θ from the following pair of equation, and describe the graph:

$$\begin{cases} x = 4 + 5\cos\theta \\ y = 3 - 5\sin\theta \end{cases}$$

Further exercises

(x1) **Ex 6G**

- Q14, 17-18

2.2 Trigonometric equations

Example 24

Solve $\tan\theta = \dfrac{1}{\sqrt{3}}$, where $0° \leq \theta \leq 360°$.

Example 25

Solve $\tan x = -3$ for $0° \leq x \leq 360°$.

Important note

- Find the equivalent acute angle first!
- Draw picture!

Example 26

Solve the following for $0° \leq x \leq 360°$:

(a) $\tan 2x = \sqrt{3}$ (b) $2\sin 3x = -1$ (c) $2\cos 2x - 1 = 0$

Important note

Check the domain!

Example 27

Solve $\sin(x - 250°) = \dfrac{\sqrt{3}}{2}$, $0° \leq x \leq 360°$.

Answer: $x = 10°, 310°$.

Example 28

Solve $5\sin^2 x = \sin x$ for $0 \leq x \leq 360°$. **Answer:** $0°, 180°, 360°, 11°32', 168°28'$

Example 29

Solve $\dfrac{4}{\cos x} - \cos x = 0,\ -180° \leq x \leq 180°$.

Example 30

Solve $\sec^2 x + \tan x = 1$, $180^\circ \leq x \leq 360^\circ$.

Answer: $180^\circ, 315^\circ, 360^\circ$

Example 31

Solve $\sin^2 x - 3\sin x \cos x + 2\cos^2 x = 0$ for $0^\circ \leq x \leq 360^\circ$.

Answer: $x = 45^\circ$, $63^\circ 26'$, 225°, $234^\circ 26'$

Example 32

If $\tan^2\theta + 2\sec^2\theta = 5$, find the exact value of $\sin^2\theta$.

Further exercises

(A) **Ex 5H**

- All questions

(X1) **Ex 6H**

- All questions

Section 3

Non right-angled trigonometry

Learning Goal(s)

Knowledge
What are the sine and cosine rules

Skills
Using the sine and cosine rules

Understanding
$\sin\theta > 0$ for $0° < \theta < 180°$ and hence a possible ambiguity with the sine rule when finding an unknown side that is opposite to the largest known angle

By the end of this section am I able to:

4.6 Establish and use the sine rule, cosine rule and the area of a triangle formula for solving problems where angles are measured in degrees, or degrees and minutes

4.7 Find angles and sides involving the ambiguous case of the sine rule. Use technology and/or geometric construction to investigate the ambiguous case of the sine rule when finding an angle, and the condition for it to arise

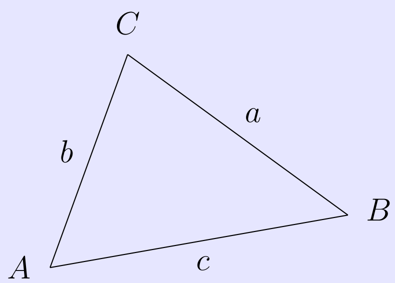
3.1 Sine rule

Definition 6

The *sine rule* (as opposed to sine ratio):

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a is opposite to angle A etc.



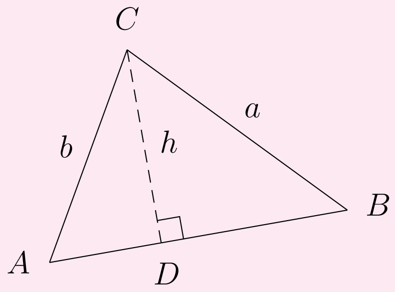
Important note

- Works on *all* triangles, not just right angled triangles.
- Use pairs of the equality.
- Use the ....reciprocal.......... to find the size of the angle.

Proof for acute angled triangles:

Steps

1. $\triangle ABC$ is any acute angled triangle. Construct perpendicular from C to AB, which will have height h:



2. Using the sine *ratio*, write the relationship between the angle A and the other two sides:

$$\frac{h}{b} = \sin A$$

$$h = b \sin A \tag{3.1}$$

3. Repeat for angle B:

$$\frac{h}{a} = \sin B$$

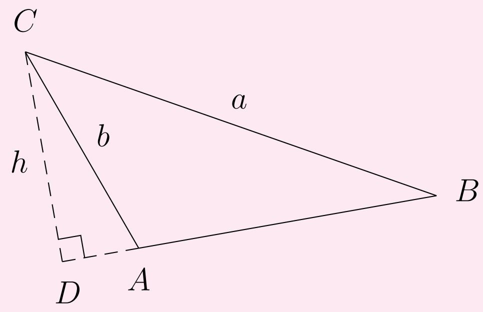
$$h = a \sin B \tag{3.2}$$

4. Equate (3.1) with (3.2), and rearrange:

Proof for obtuse angled triangles:

Steps

1. $\triangle ABC$ is any obtuse angled triangle, obtuse at A. Construct perpendicular from C to AB (extended to D), which will have height h:



2. Using the sine *ratio*, write the relationship between the angle $(180° - A)$ and the other two sides:

$$\frac{h}{b} = \sin(180° - A)$$

$$h = b \sin A \qquad (3.3)$$

3. Repeat for angle B:

$$\frac{h}{a} = \sin B$$

$$h = a \sin B \qquad (3.4)$$

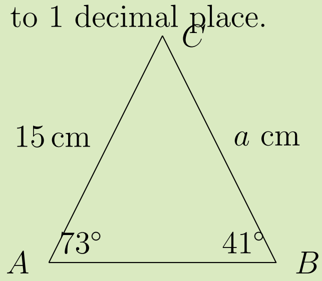
4. Equate (3.3) with (3.4), and rearrange:

Laws/Results

- The ......longest...... side will be opposite to the ......largest...... angle.
- The ......shortest...... side will be opposite to the ......smallest...... angle.

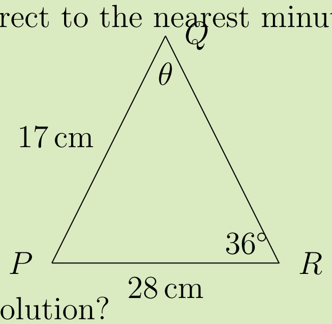
Example 33

Find the value of a, correct to 1 decimal place. **Answer:** 21.9



Example 34

⚠ Find the value of θ, correct to the nearest minute. **Answer:** $75^\circ 30'$



Is there a second possible solution?

Example 35

A plane flew 160 km from P to Q on a bearing of 200°. It then turned and flew 225 km to R, which is due west of P. Find, correct to the nearest degree, the bearing of Q from R.

Answer: 132°

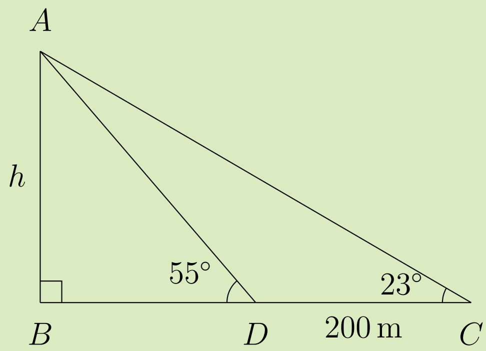
Example 36

In the diagram, $\angle ADB = 55°$, $\angle ACD = 23°$, $DC = 200\,\text{m}$. Let $AB = h$.

(a) Use the sine rule in $\triangle ADC$ to show that

$$AD = \frac{200\sin 23°}{\sin 32°}$$

(b) Hence show that $h = \dfrac{200\sin 23° \sin 55°}{\sin 32°}$, and evaluate h.

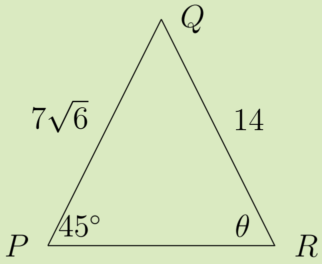


3.1.1 Ambiguous case

- Occurs when an unknown angle is ....opposite........ to the ....longer........ side, with the sum of angles being less than 180°.

Example 37

Find the value of θ.



Example 38

In $\triangle ABC$, $\angle A = 25°$, $BC = 9\,\text{cm}$ and $AB = 20\,\text{cm}$. Find the possible value(s) of $\angle C$, correct to the nearest degree and hence show there are two possible triangles.

Answer: $70°, 110°$

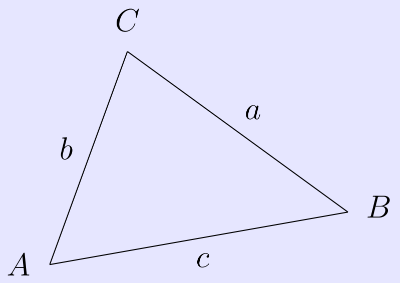
3.2 Area via sine rule

Definition 7

The *area of a triangle via sine rule*:

$$A = \frac{1}{2}ab\sin C \tag{3.5}$$

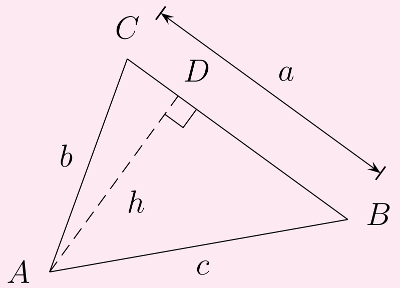
where $\angle C$ is the included angle between side lengths a and b.



Proof (valid for acute or obtuse-angled triangles)

Steps

1. $\triangle ABC$ is any triangle. Construct perpendicular from A to BC, which will have height h:



2. In $\triangle ADC$, use the sine *ratio* to write the relationship between the angle C and the other two sides:

$$\frac{h}{b} = \sin C$$

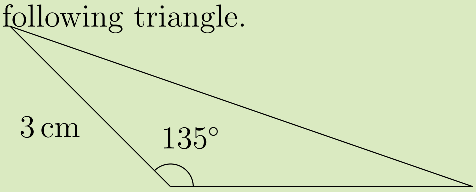
$$\therefore \quad h = b\sin C \tag{3.6}$$

3. In $\triangle ABC$, $A = \dfrac{1}{2} \times$ base $\times$ height :

$$\begin{aligned} A &= \frac{1}{2} \times a \times h \\ &= \frac{1}{2} \times a \times b\sin C \quad \text{[substitute (3.6)]} \qquad (3.7) \\ &= \frac{1}{2}ab\sin C \end{aligned}$$

Example 39

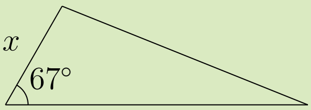
Find the exact area of the following triangle. **Answer:** $3\sqrt{2}$



Example 40

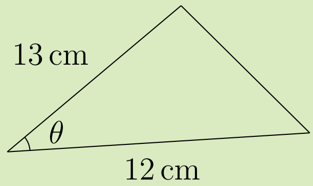
Given an area of $72\,\text{cm}^2$, find the value of x correct to 4 significant figures.

Answer: 6.518



Example 41

Find the value of θ, correct to the nearest minute given the area of the triangle has area $60\,\text{cm}^2$.

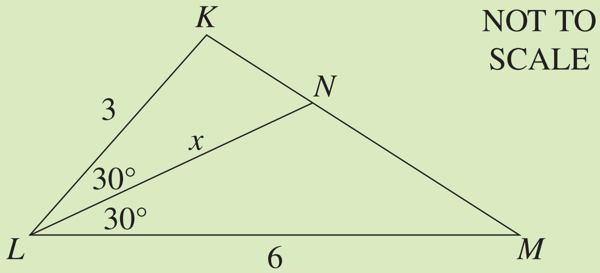


Important note

⚠ Beware of the ambiguous case!

Example 42

[2018 2U HSC] In $\triangle KLM$, KL has length 3, LM has length 6 and $\angle KLM$ is 60°. The point N is chosen on side KM so that LN bisects $\angle KLM$. The length LN is x.



i. Find the exact value of the area of $\triangle KLM$. **1**

ii. Hence, or otherwise, find the exact value of x. **2**

Further exercises

(A) Ex 5I
- Q1-4 last 2 columns
- Q5-17

(x1) Ex 6I
- Q1-4 last column
- Q9-17, 21-22

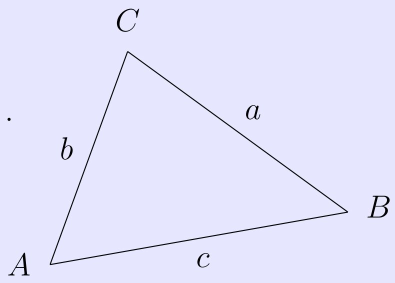
3.3 Cosine rule

Definition 8

The *cosine rule* (as opposed to cosine ratio):

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad (3.8)$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad (3.9)$$



where a is opposite to angle A etc.

Important note

- Works on *all* triangles, not just right angled triangles.
- Used to find
  - an angle when all side lengths are known (Equation (3.9)), or
  - a side length, when two other side lengths and the included angle are known (Equation (3.8)).

Example 43

Without using a calculator, find the value of

(a) z if in $\triangle XYZ$, $x = 2$, $y = 5$ and $\cos Z = \frac{4}{5}$.

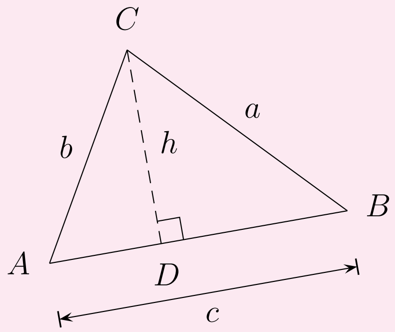
(b) $\cos B$ if in $\triangle BCD$, $b = 5$, $c = 6$ and $d = 7$.

Answer: (a) $\sqrt{13}$ (b) $\frac{5}{7}$

Proof for acute angled triangles:

Steps

1. $\triangle ABC$ is any acute angled triangle. Construct perpendicular from C to AB, which will have height h, and let $AD = x$.



2. Write a relationship between BD, CD and BC:

$$a^2 = h^2 + (c - x)^2 \tag{3.10}$$

3. Write a relationship between CD, CA and AD:

$$b^2 = h^2 + x^2 \tag{3.11}$$

4. Substitute (3.11) into (3.10), then expand and simplify:

5. Write a relationship between $\angle A$, AD and AC:

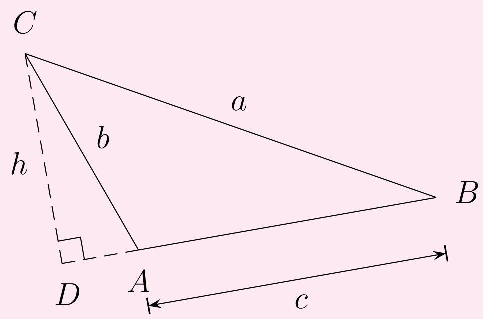
$$\frac{x}{b} = \cos A \tag{3.12}$$

6. Substitute (3.12) into previous result:

Proof for obtuse angled triangles:

Steps

1. $\triangle ABC$ is any obtuse angled triangle. Construct perpendicular from C to AB (AB requires extension), which will have height h, and let $AD = x$.



2. Write a relationship between BD, CD and BC:

$$a^2 = h^2 + (c+x)^2 \tag{3.13}$$

3. Write a relationship between CD, CA and AD:

$$b^2 = h^2 + x^2 \tag{3.14}$$

4. Substitute (3.14) into (3.13), then expand and simplify:

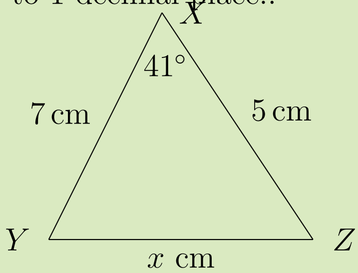
5. Write a relationship between $\angle DAC$, AD and AC:

$$\frac{x}{b} = \cos(180^\circ - A) \tag{3.15}$$

6. Substitute (3.15) into previous result:

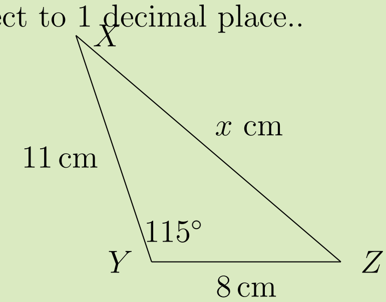
Example 44

Find the value of x, correct to 1 decimal place.. **Answer:** 4.6



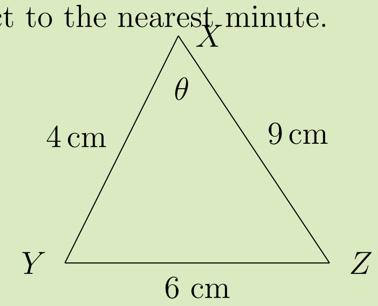
Example 45

Find the value of x, correct to 1 decimal place.. **Answer:** 16.1



Example 46

Find the value of θ, correct to the nearest minute. **Answer:** $32°5'$



Example 47

Find the size of the smallest angle in a triangle with side lengths 15 cm, 11 cm and 8 cm. **Answer:** $31°17'$

Example 48

Alana drove 42 km from E to F on a bearing of 345°. She then turned and drove 73 km on a bearing of 240° to G. Find the distance EG, correct to 1 decimal place.

Answer: 74.2 km

Important note

Draw picture!

Example 49

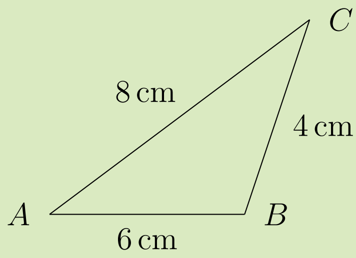
(a) If $\cos\alpha = \frac{5}{16}$, find the exact value of $\sin\alpha$, given $0° < \alpha < 90°$.

(b) The sides of a triangular field have lengths 80 m, 90 m, 100 m. Calculate the exact area of the field.

Answer: $225\sqrt{231}$

Example 50

[2015 HSC Q13] The diagram shows $\triangle ABC$ with sides $AB = 6\,\text{cm}$, $BC = 4\,\text{cm}$ and $AC = 8\,\text{cm}$.



(i) Show that $\cos A = \dfrac{7}{8}$. **1**

(ii) By finding the exact value of $\sin A$, determine the exact value of the area of $\triangle ABC$. **2**

Answer: $3\sqrt{15}$

Further exercises

(A) Ex 5J

- Q1-2 last 2 columns
- Q3-14

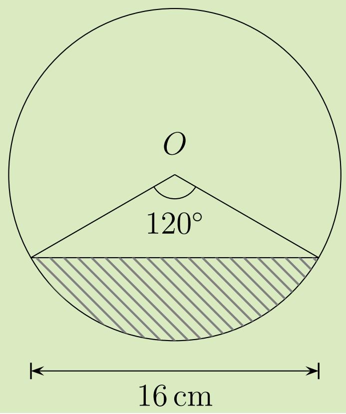
(x1) Ex 6J

- Q1-2 last column
- Q3-12, 14-17

3.4 General problem solving

Example 51

The surface of the water in a horizontal pipe is 16 cm wide and subtends an angle of 120° at the centre of the pipe, as shown.



Find, as an exact value:

(a) The distance from the centre of the pipe to the water surface.

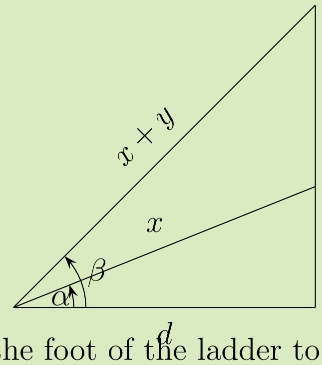
(b) The diameter of the pipe.

(c) The maximum depth of the water.

Answer: (a) $\frac{8}{\sqrt{3}}$ (b) $\frac{32}{\sqrt{3}}$ (c) $\frac{8}{\sqrt{3}}$

Example 52

[Ex 4J Q17] A ladder of length x cm is inclined at an angle α to the ground. The foot of the ladder is fixed. If the ladder were y cm longer, the inclination to the horizontal would be β.



Show that the distance from the foot of the ladder to the wall is given by

$$\frac{y\cos\alpha\cos\beta}{\cos\alpha-\cos\beta}\text{ cm}$$

Further exercises

Ⓐ **Ex 5K**

- Q1-12, 15-18

(x1) **Ex 6K**

- Q4-12, 15-19

Note: Do *not* do all of this set in one sitting! Spread it out over a week!

Section 4

3D Trigonometry

Learning Goal(s)

Knowledge
Using right angled triangles, non right angled triangles and bearings to assist with problem solving

Skills
Splitting particular planes into triangles for ease of calculation

Understanding
Some right angles may look awkward when drawn on a 2D sheet of 'paper'

By the end of this section am I able to:

4.9 Solve problems involving the use of trigonometry in two and three dimensions

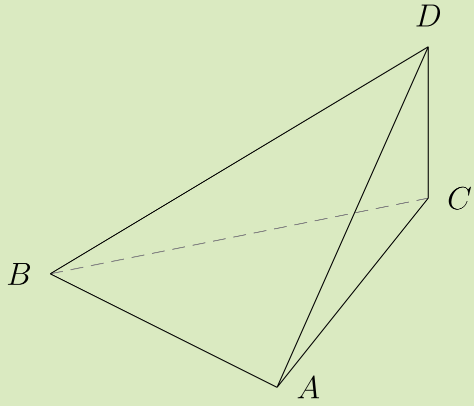
4.1 Techniques required

- Reproduce diagram on to working paper.
- Mark on your own diagram, the measurements provided.
- Where necessary, draw NSEW axes for bearings.
- Use sine/cosine rules sparingly and only when necessary. (Right angled trigonometry)
- Look for
  - Right angles
  - Complementary/supplementary angles
  - Isosceles/equilateral triangles

4.2 Examples

Example 53

[2009 NSBHS Ext 1 Assessment Task 2] The points A and B are 500 m apart on the ground and D is the top of a tower. $\angle BAD$ and $\angle DBA$ are 59° and 54° respectively. The elevation of D from A is 5°.



Copy the diagram into your writing booklet and mark on the figure all the angles stated above.

(a) Show that the height h metres of the tower is given by **3**

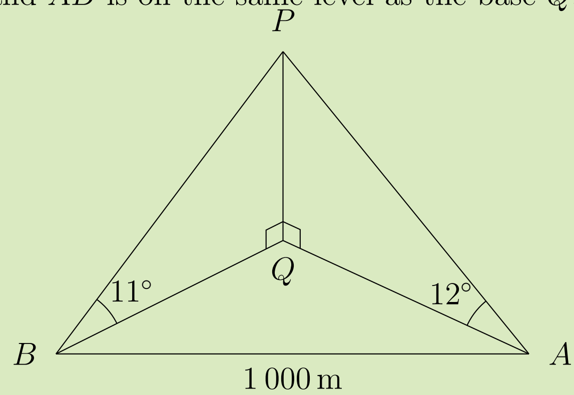
$$h = \frac{500 \sin 5° \sin 54°}{\sin 67°}$$

(b) Find h to the nearest metre. **1**

Example 54

[2011 NSBHS Ext 1 Assessment Task 2] The angle of elevation of a tower PQ of height h metres from a point A due east of Q is $12°$. From another point B, the bearing of the tower is $051°$T and the angle of elevation is $11°$.

$AB = 1\,000$ m and AB is on the same level as the base Q.



(a) Show that $\angle AQB = 141°$. **1**

(b) Show that **3**

$$h^2 = \frac{1\,000\,000}{\tan^2 78° + \tan^2 79° - 2\tan 78° \tan 79° \cos 141°}$$

(c) Find h, correct to the nearest metre. **1**

Important note

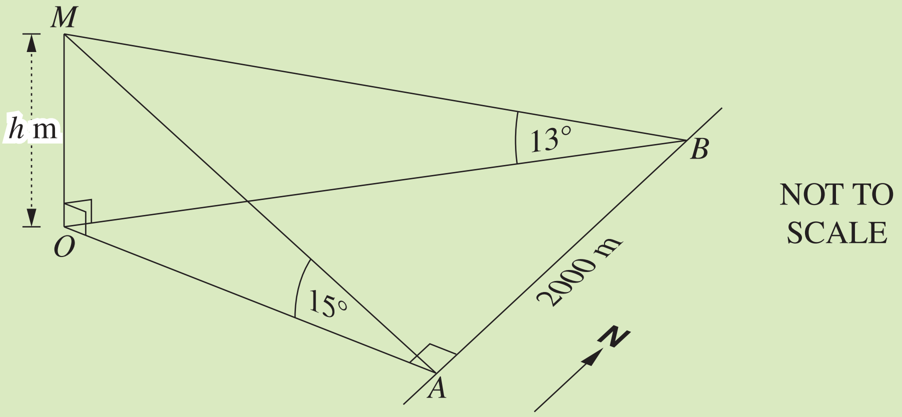
⚠ The picture may need to be redrawn!

Example 55

[2015 Ext 1 HSC Q12] A person walks 2 000 metres due north along a road from point A to point B. The point A is due east of a mountain OM, where M is the top of the mountain. The point O is directly below point M and is on the same horizontal plane as the road. The height of the mountain above point O is h metres.

From point A, the angle of elevation to the top of the mountain is 15°.

From point B, the angle of elevation to the top of the mountain is 13°.



i. Show that $OA = h \cot 15°$. **1**

ii. Hence, find the value of h. **2**

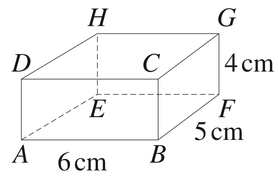
Further exercises (Legacy Textbooks)

Ex 2G/2H (Pender, Sadler, Shea, & Ward, 2000) (see next page)

- All questions

Exercise 2G

1. The diagram shows a box in the shape of a rectangular prism.

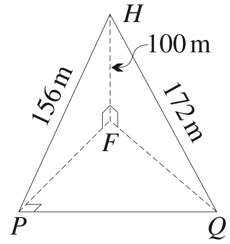


(a) Find, correct to the nearest minute, the angle that the diagonal plane $AEGC$ makes with the face $BCGF$.

(b) Find the length of the diagonal AG of the box, correct to the nearest millimetre.

(c) Find, correct to the nearest minute, the angle that the diagonal AG makes with the base $AEFB$.

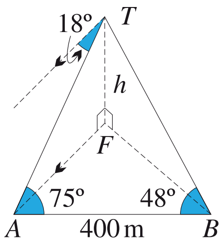
2. A helicopter H is hovering 100 metres above the level ground below. Two observers P and Q on the ground are 156 metres and 172 metres respectively from H. The helicopter is due north of P, while Q is due east of P.



(a) Find the angles of elevation of the helicopter from P and Q, correct to the nearest minute,

(b) Find the distance between the two observers P and Q, correct to the nearest metre.

3. The points A and B are 400 metres apart in a horizontal plane. The angle of depression of A from the top T of a vertical tower standing on the plane is $18°$. Also, $\angle TAB = 75°$ and $\angle TBA = 48°$.

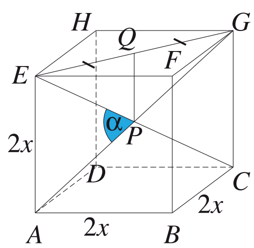


(a) Show that $TA = \dfrac{400 \sin 48°}{\sin 57°}$.

(b) Hence find the height h of the tower, correct to the nearest metre.

(c) Find, correct to the nearest degree, the angle of depression of B from T.

4. The diagram shows a cube $ABCDEFGH$. The diagonals AG and CE meet at P. Q is the midpoint of the diagonal EG of the top face. Suppose that $2x$ is the side length of the cube and α is the acute angle between the diagonals AG and CE.



(a) State the length of PQ.

(b) Show that $EQ = \sqrt{2}\,x$.

(c) Hence show that $EP = \sqrt{3}\,x$.

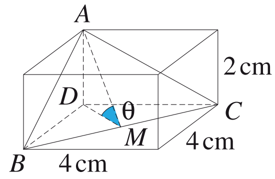
(d) Hence show that $\cos \angle EPQ = \frac{1}{3}\sqrt{3}$.

(e) By using an appropriate double-angle formula, deduce that $\cos \angle EPG = -\frac{1}{3}$, and hence that $\cos \alpha = \frac{1}{3}$.

(f) Confirm the fact that $\cos \alpha = \frac{1}{3}$ by using the cosine rule in $\triangle APE$.

(g) Find, correct to the nearest minute, the angle that the diagonal AG makes with the base $ABCD$ of the cube.

5. The prism in the diagram has a square base of side 4 cm and its height is 2 cm. ABC is a diagonal plane of the prism. Let θ be the acute angle between the diagonal plane and the base of the prism.

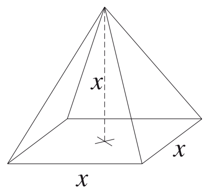


(a) Show that $MD = 2\sqrt{2}$ cm.

(b) Hence find θ, correct to the nearest minute.

DEVELOPMENT

6. The diagram shows a square pyramid whose perpendicular height is equal to the side of the base. Find, correct to the nearest minute:

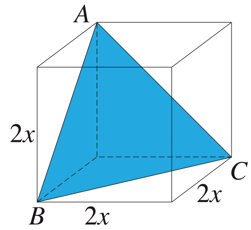


(a) the angle between an oblique face and the base,

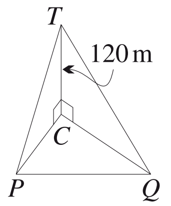
(b) the angle between a slant edge and the base,

(c) the angle between an opposite pair of oblique faces.

7. The diagram shows a cube of side $2x$ in which a diagonal plane ABC is drawn. Find, correct to the nearest minute, the angle between this diagonal plane and the base of the cube.



8. Two boats P and Q are observed from the top T of a vertical cliff CT of height 120 metres. P is on a bearing of 195° from the cliff and its angle of depression from T is 22°. Q is on a bearing of 161° from the cliff and its angle of depression from T is 27°.

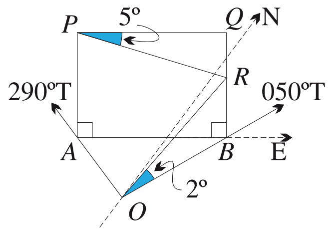


(a) Show that $\angle PCQ = 34°$.

(b) Use the cosine rule to show that the boats are approximately 166 metres apart.

9. A plane is flying along the path PR. Its constant speed is 300 km/h. It flies directly over landmarks A and B, where B is due east of A. An observer at O first sights the plane when it is over A at a bearing of 290° T, and then, ten minutes later, he sights the plane when it is over B at a bearing of 50° T and with an angle of elevation of 2°.

(a) Show that the plane has travelled 50 km in the ten minutes between observations.

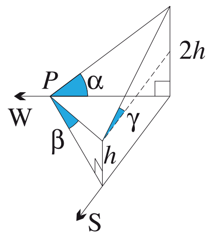


(b) Show that $\angle AOB = 120°$.

(c) Prove that the observer is 19 670 metres, correct to the nearest ten metres, from landmark B.

(d) Find the height h of the plane, correct to the nearest 10 metres, when it was directly above A.

10. Two towers of height $2h$ and h stand on a horizontal plane. The shorter tower is due south of the taller tower. From a point P due west of the taller tower, the angles of elevation of the tops of the taller and shorter towers are α and β respectively. The angle of elevation of the top of the taller tower from the top of the shorter tower is γ. Show that



$$4\cot^2\alpha = \cot^2\beta - \cot^2\gamma.$$

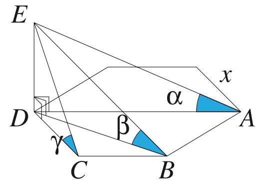
11. A, B, C and D are four of the vertices of a horizontal regular hexagon of side length x. DE is vertical and subtends angles of α, β and γ at A, B and C respectively.

(a) Show that each interior angle of a regular hexagon is 120°.

(b) Show that $\angle BAD = 60°$ and $\angle ABD = 90°$.

(c) Show that $BD = \sqrt{3}\,x$ and $AD = 2x$.

(d) Hence show that $\cot^2\alpha = \cot^2\beta + \cot^2\gamma$.

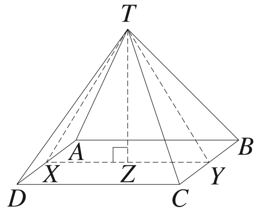


12. The diagram shows a rectangular pyramid. X and Y are the midpoints of AD and BC respectively and T is directly above Z. $TX = 15$ cm, $TY = 20$ cm, $AB = 25$ cm and $BC = 10$ cm.

(a) Show that $\angle XTY = 90°$.

(b) Hence show that T is 12 cm above the base.

(c) Hence find, correct to the nearest minute, the angle that the front face DCT makes with the base.



13. A plane is flying due east at 600 km/h at a constant altitude. From an observation point P on the ground, the plane is sighted on a bearing of 320°. One minute later, the bearing of the plane is 75° and its angle of elevation is 25°.

(a) How far has the plane travelled between the two sightings?

(b) Draw a diagram to represent the given information.

(c) Show that the altitude h metres of the plane is given by $h = \dfrac{10\,000 \sin 50° \tan 25°}{\sin 65°}$, and hence find the altitude, correct to the nearest metre.

(d) Find, correct to the nearest degree, the angle of elevation of the plane from P when it was first sighted.

14. (a) The diagonal PQ of the rectangular prism in the diagram makes angles of α, β and γ respectively with the edges PA, PB and PC.

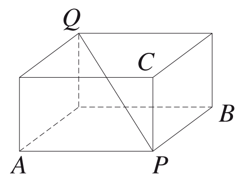
(i) Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(ii) What is the two-dimensional version of this result?

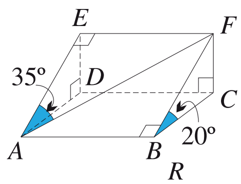
(b) Suppose that the diagonal PQ makes angles of θ, ϕ and ψ with the three faces of the prism that meet at P.

(i) Prove that $\sin^2\theta + \sin^2\phi + \sin^2\psi = 1$.

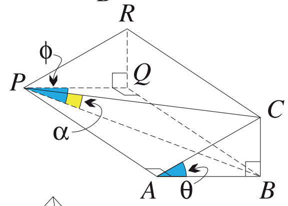
(ii) What is the two-dimensional version of this result?



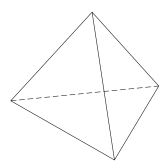
15. The diagram shows a hill inclined at 20° to the horizontal. A straight road AF on the hill makes an angle of 35° with a line of greatest slope. Find, correct to the nearest minute, the inclination of the road to the horizontal.



16. The plane surface $APRC$ is inclined at an angle θ to the horizontal plane $APQB$. Both $APRC$ and $APQB$ are rectangles. PR is a line of greatest slope on the inclined plane. $\angle BPQ = \phi$ and $\angle BPC = \alpha$. Show that $\tan\alpha = \tan\theta\cos\phi$.



17. The diagram shows a triangular pyramid, all of whose faces are equilateral triangles — such a solid is called a *regular tetrahedron*. Suppose that the slant edges are inclined at an angle θ to the base. Show that $\cos\theta = \frac{1}{3}\sqrt{3}$.



18. A square pyramid has perpendicular height equal to the side length of its base.

(a) Show that the angle between a slant edge and a base edge it meets is $\cos^{-1}\frac{1}{6}\sqrt{6}$.

(b) Show that the angle between adjacent oblique faces is $\cos^{-1}(-\frac{1}{5})$.

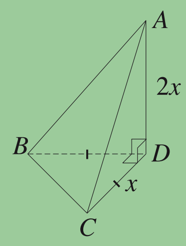
EXTENSION

19. A cube has one edge AB of its base inclined at an angle θ to the horizontal and another edge AC of its base horizontal. The diagonal AP of the cube is inclined at angle ϕ to the horizontal.

(a) Show that the height h of the point P above the horizontal plane containing the edge AC is given by $h = x\cos\theta(1+\tan\theta)$, where x is the side length of the cube.

(b) Hence show that $\cos^2\phi = \frac{2}{3}(1-\sin\theta\cos\theta)$.

20. The diagram shows a triangular pyramid $ABCD$. The horizontal base BCD is an isosceles triangle whose equal sides BD and CD are at right angles and have length x units. The edge AD has length $2x$ units and is vertical.

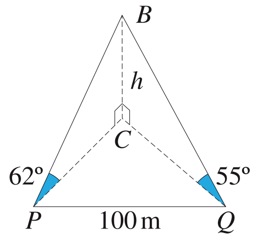


(a) Let α be the acute angle between the front face ABC and the base BCD. Show that $\alpha = \cos^{-1}\frac{1}{3}$.

(b) Let θ be the acute angle between the front face ABC and a side face (that is, either ABD or ACD). Show that $\theta = \cos^{-1}\frac{2}{3}$.

Exercise 2H

1. A balloon B is due north of an observer P and its angle of elevation is 62°. From another observer Q 100 metres from P, the balloon is due west and its angle of elevation is 55°. Let the height of the balloon be h metres and let C be the point on the level ground vertically below B.



(a) Show that $PC = h\cot 62^\circ$, and write down a similar expression for QC.

(b) Explain why $\angle PCQ = 90^\circ$.

(c) Use Pythagoras' theorem in $\triangle CPQ$ to show that

$$h^2 = \frac{100^2}{\cot^2 62^\circ + \cot^2 55^\circ}.$$

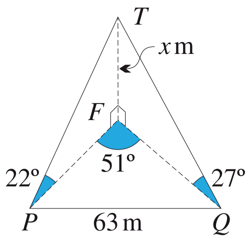
(d) Hence find h, correct to the nearest metre.

2. From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20°. From a point Q situated 40 metres from P and due east of the tower, the angle of elevation is 35°. Let h metres be the height of the tower.

(a) Draw a diagram to represent the situation.

(b) Show that $h = \dfrac{40}{\sqrt{\tan^2 70^\circ + \tan^2 55^\circ}}$, and evaluate h, correct to the nearest metre.

3. In the diagram, TF represents a vertical tower of height x metres standing on level ground. From P and Q at ground level, the angles of elevation of T are 22° and 27° respectively. $PQ = 63$ metres and $\angle PFQ = 51^\circ$.

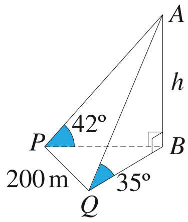


(a) Show that $PF = x \cot 22^\circ$ and write down a similar expression for QF.

(b) Use the cosine rule to show that $x^2 = \dfrac{63^2}{\cot^2 22^\circ + \cot^2 27^\circ - 2\cot 22^\circ \cot 27^\circ \cos 51^\circ}$.

(c) Use a calculator to show that $x \doteqdot 32$.

4. The points P, Q and B lie in a horizontal plane. From P, which is due west of B, the angle of elevation of the top of a tower AB of height h metres is 42°. From Q, which is on a bearing of 196° from the tower, the angle of elevation of the top of the tower is 35°. The distance PQ is 200 metres.



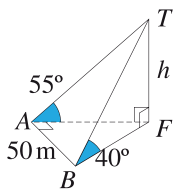
(a) Explain why $\angle PBQ = 74^\circ$.

(b) Show that $h^2 = \dfrac{200^2}{\cot^2 42^\circ + \cot^2 35^\circ - 2\cot 35^\circ \cot 42^\circ \cos 74^\circ}$.

(c) Hence find the height of the tower, correct to the nearest metre.

DEVELOPMENT

5. The diagram shows a tower of height h metres standing on level ground. The angles of elevation of the top T of the tower from two points A and B on the ground nearby are 55° and 40° respectively. The distance AB is 50 metres and the interval AB is perpendicular to the interval AF, where F is the foot of the tower.



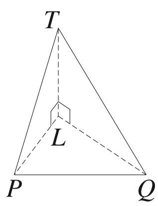
(a) Find AT and BT in terms of h.

(b) What is the size of $\angle BAT$?

(c) Use Pythagoras' theorem in $\triangle BAT$ to show that $h = \dfrac{50 \sin 55^\circ \sin 40^\circ}{\sqrt{\sin^2 55^\circ - \sin^2 40^\circ}}$.

(d) Hence find the height of the tower, correct to the nearest metre.

6. The diagram shows two observers P and Q 600 metres apart on level ground. The angles of elevation of the top T of a landmark TL from P and Q are 9° and 12° respectively. The bearings of the landmark from P and Q are 32° and 306° respectively. Let $h = TL$ be the height of the landmark.



(a) Show that $\angle PLQ = 86^\circ$.

(b) Find expressions for PL and QL in terms of h.

(c) Hence show that $h \doteqdot 79$ metres.

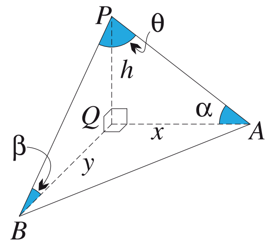
7. PQ is a straight level road. Q is x metres due east of P. A vertical tower of height h metres is situated due north of P. The angles of elevation of the top of the tower from P and Q are α and β respectively.

(a) Draw a diagram representing the situation.

(b) Show that $x^2 + h^2 \cot^2 \alpha = h^2 \cot^2 \beta$.

(c) Hence show that $h = \dfrac{x \sin\alpha \sin\beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}$.

8. In the diagram of a triangular pyramid, $AQ = x$, $BQ = y$, $PQ = h$, $\angle APB = \theta$, $\angle PAQ = \alpha$ and $\angle PBQ = \beta$. Also, there are three right angles at Q.



(a) Show that $x = h \cot \alpha$ and write down a similar expression for y.

(b) Use Pythagoras' theorem and the cosine rule to show that $\cos \theta = \dfrac{h^2}{\sqrt{(x^2 + h^2)(y^2 + h^2)}}$.

(c) Hence show that $\sin \alpha \sin \beta = \cos \theta$.

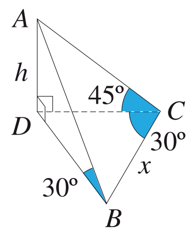
9. A man walking along a straight, flat road passes by three observation points P, Q and R at intervals of 200 metres. From these three points, the respective angles of elevation of the top of a vertical tower are 30°, 45° and 45°. Let h metres be the height of the tower.

(a) Draw a diagram representing the situation.

(b) (i) Find, in terms of h, the distances from P, Q and R to the foot F of the tower.

(ii) Let $\angle FRQ = \alpha$. Find two different expressions for $\cos \alpha$ in terms of h, and hence find the height of the tower.

10. $ABCD$ is a triangular pyramid with base BCD and perpendicular height AD.



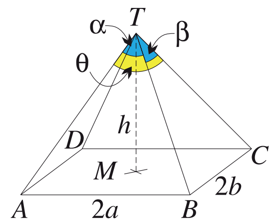
(a) Find BD and CD in terms of h.

(b) Use the cosine rule to show that $2h^2 = x^2 - \sqrt{3}\,hx$.

(c) Let $u = \dfrac{h}{x}$. Write the result of the previous part as a quadratic equation in u, and hence show that

$$\frac{h}{x} = \frac{\sqrt{11} - \sqrt{3}}{4}.$$

11. The diagram shows a rectangular pyramid. The base $ABCD$ has sides $2a$ and $2b$ and its diagonals meet at M. The perpendicular height TM is h. Let $\angle ATB = \alpha$, $\angle BTC = \beta$ and $\angle ATC = \theta$.

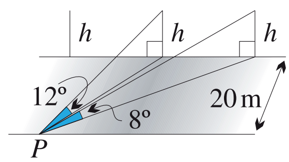


(a) Use Pythagoras' theorem to find AC, AM and AT in terms of a, b and h.

(b) Use the cosine rule to find $\cos \alpha$, $\cos \beta$ and $\cos \theta$ in terms of a, b and h.

(c) Show that $\cos \alpha + \cos \beta = 1 + \cos \theta$.

12. The diagram shows three telegraph poles of equal height h metres standing equally spaced on the same side of a straight road 20 metres wide. From an observer at P on the other side of the road directly opposite the first pole, the angles of elevation of the tops of the other two poles are 12° and 8° respectively. Let x metres be the distance between two adjacent poles.

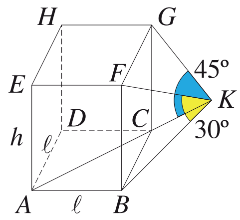


(a) Show that $h^2 = \dfrac{x^2 + 20^2}{\cot^2 12^\circ}$.

(b) Hence show that $x^2 = \dfrac{20^2(\cot^2 8^\circ - \cot^2 12^\circ)}{4\cot^2 12^\circ - \cot^2 8^\circ}$.

(c) Hence calculate the distance between adjacent poles, correct to the nearest metre.

13. A building is in the shape of a square prism with base edge ℓ metres and height h metres. It stands on level ground. The diagonal AC of the base is extended to K, and from K, the respective angles of elevation of F and G are 30° and 45°.



(a) Show that $BK^2 = h^2 + \ell^2 + \sqrt{2}\,h\ell$.

(b) Hence show that $2h^2 - \ell^2 = \sqrt{2}\,h\ell$.

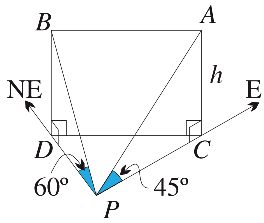
(c) Deduce that $\dfrac{h}{\ell} = \dfrac{\sqrt{2}+\sqrt{10}}{4}$.

14. From a point P on level ground, a man observes the angle of elevation of the summit of a mountain due north of him to be 18°. After walking 3 km in a direction N50°E to a point Q, the man finds that the angle of elevation of the summit is now 13°.

(a) Show that $(\cot^2 13^\circ - \cot^2 18^\circ)h^2 + (6000 \cot 18^\circ \cos 50^\circ)h - 3000^2 = 0$, where h metres is the height of the mountain.

(b) Hence find the height, correct to the nearest metre.

15. A plane is flying at a constant height h, and with constant speed. An observer at P sighted the plane due east at an angle of elevation of 45°. Soon after it was sighted again in a north-easterly direction at an angle of elevation of 60°.



(a) Write down expressions for PC and PD in terms of h.

(b) Show that $CD^2 = \frac{1}{3}h^2(4 - \sqrt{6})$.

(c) Find, as a bearing correct to the nearest degree, the direction in which the plane is flying.

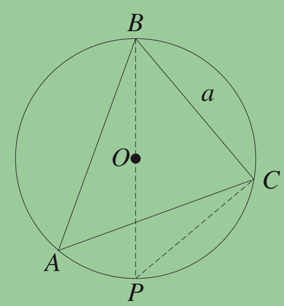
16. Three tourists T_1, T_2 and T_3 at ground level are observing a landmark L. T_1 is due north of L, T_3 is due east of L, and T_2 is on the line of sight from T_1 to T_3 and between them. The angles of elevation to the top of L from T_1, T_2 and T_3 are 25°, 32° and 36° respectively.

(a) Show that $\tan \angle LT_1T_2 = \dfrac{\cot 36^\circ}{\cot 25^\circ}$.

(b) Use the sine rule in $\triangle LT_1T_2$ to find, correct to the nearest minute, the bearing of T_2 from L.

EXTENSION

17. (a) Use the diagram on the right to show that the diameter BP of the circumcircle of $\triangle ABC$ is $\dfrac{a}{\sin A}$.



(b) A vertical tower stands on level ground. From three observation points P, Q and R on the ground, the top of the tower has the same angle of elevation of 30°. The distances PQ, PR and QR are 60 metres, 50 metres and 40 metres respectively.

(i) Explain why the foot of the tower is the centre of the circumcircle of $\triangle PQR$.

(ii) Use the result in part (a) to show that the height of the tower is $\frac{80}{21}\sqrt{21}$ metres.

Exercise **2G** (Page 70)

1(a) $56°19'$ **(b)** $8{\cdot}8$ cm **(c)** $27°7'$

2(a) $39°52'$, $35°33'$ **(b)** 72 metres

3(b) 110 metres **(c)** $14°$

4(a) x **(g)** $35°16'$

5(b) $35°16'$

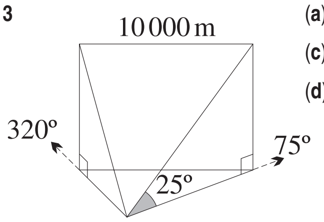
6(a) $63°26'$ **(b)** $54°44'$ **(c)** $53°8'$

7 $54°44'$

9(d) 5040 metres

12(c) $67°23'$

13



(a) 10 000 metres
(c) 3941 metres
(d) $54°$

14(a)(ii) $\cos^2\alpha + \cos^2\beta = 1$, where $\alpha + \beta = 90°$.
(b)(ii) $\sin^2\theta + \sin^2\phi = 1$, where $\theta + \phi = 90°$.

15 $16°16'$

Exercise **2H** (Page 75)

1(a) $h\cot 55°$

(b) It is the angle between south and east.

(d) 114 metres

2(b) 13 metres

3(a) $x\cot 27°$

4(c) 129 metres

5(a) $AT = h\operatorname{cosec} 55°,\ BT = h\operatorname{cosec} 40°$ **(b)** $90°$

(d) 52 metres

6(b) $PL = h\cot 9°$, $QL = h\cot 12°$

8(a) $y = h\cot\beta$

9(b)(i) $\sqrt{3}\,h,\ h,\ h$

(ii) $\cos\alpha = \frac{100}{h}$ or $\frac{80\,000 - h^2}{400h}$, $h = 200$ metres

10(a) $BD = \sqrt{3}\,h$, $CD = h$

11(a) $AC = 2\sqrt{a^2+b^2}\,,\ AM = \sqrt{a^2+b^2}\,,$
$AT = \sqrt{a^2+b^2+h^2}$ **(b)** $\cos\alpha = \frac{-a^2+b^2+h^2}{a^2+b^2+h^2}$,
$\cos\beta = \frac{a^2-b^2+h^2}{a^2+b^2+h^2}$, $\cos\theta = \frac{-a^2-b^2+h^2}{a^2+b^2+h^2}$

12(c) 17 metres

14(b) 535 metres

15(a) $PC = h,\ PD = \frac{1}{3}h\sqrt{3}$ **(c)** $305°$

16(b) $13°41'$

17(b)(i) The foot of the tower is equidistant from P, Q and R, the distance being $h\cot 30°$.

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$

$S = \frac{a}{1 - r}, \ |r| < 1$

Logarithmic and Exponential Functions

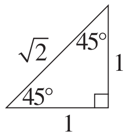
$\log_a a^x = x = a^{\log_a x}$

$\log_a x = \frac{\log_b x}{\log_b a}$

$a^x = e^{x \ln a}$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

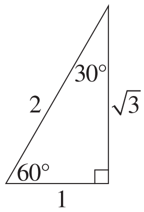


$A = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \frac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

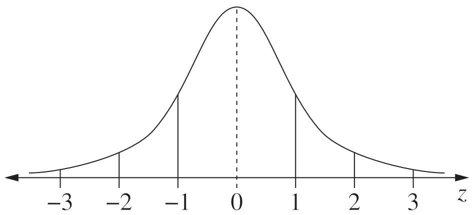
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

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